

Reverse Poincaré inequalities, Isoperimetry, and Riesz transforms in Carnot groups

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Abstract

We prove an optimal reverse Poincaré inequality for the heat semigroup generated by the sub-Laplacian on a Carnot group of any step. As an application we give new proofs of the isoperimetric inequality and of the boundedness of the Riesz transform in Carnot groups.

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1 Introduction

In this paper, we are interested in reverse Poincaré inequalities for Markov semigroups. If P_t is a Markov semigroup generated by a diffusion operator L , such an inequality writes in the form of

$$\Gamma(P_t f) \leq C(t)(P_t f^2 - (P_t f)^2) \quad (1.1)$$

where $C(t)$ is a positive constant and Γ the *carré du champ* operator:

$$\Gamma(f) = \frac{1}{2}(Lf^2 - 2fLf).$$

This inequality is a regularization estimate in the sense that it allows to control derivatives of $P_t f$ in terms of the sup-norm of f only. If L is the Laplace operator on \mathbb{R}^n , it is easy to check that the inequality (1.1) holds with $C(t) = \frac{1}{2t}$, and this is the best possible constant.

The reverse Poincaré inequality is closely related to intrinsic curvature properties of the generator L . Actually, it turns out that the inequality

$$\Gamma(P_t f) \leq \frac{K}{e^{2Kt} - 1} (P_t f^2 - (P_t f)^2)$$

is equivalent to the Bakry-Émery criterion

$$\Gamma_2(f) \geq K\Gamma(f),$$

where

$$\Gamma_2(f) = \frac{1}{2}(L\Gamma(f) - 2\Gamma(f, Lf)),$$

is the usual Bakry's Γ_2 operator. For further details on this, we refer to the book by Bakry-Gentil-Ledoux [3]. As an example, on a complete Riemannian manifold, the inequality

$$\|\nabla P_t f\|^2 \leq \frac{K}{e^{2Kt} - 1} (P_t f^2 - (P_t f)^2)$$

is satisfied for the semigroup generated by the Laplace-Beltrami operator, if and only if the Ricci curvature of the manifold is bounded from below by K .

As is it now understood, the Bakry-Émery criterion requires some form of ellipticity for the generator and typically fails to hold for strictly hypoelliptic operators [2, 4]. However, if L is a hypoelliptic diffusion operator, the semigroup P_t is smoothing in the sense that it transforms bounded Borel functions into smooth functions. For this reason, it is natural to expect, that under suitable conditions an inequality like (1.1) should hold. In [2], it was proved that if L is the sub-Laplacian in the Heisenberg group, the following reverse Poincaré inequality holds

$$\Gamma(P_t f) \leq \frac{1}{t} (P_t f^2 - (P_t f)^2).$$

Moreover, the constant $\frac{1}{t}$ is optimal. In the same reference, it was conjectured that a similar inequality should hold on any Carnot group. In the present paper, we prove that this is indeed the case and moreover compute the optimal constant of the inequality.

The paper is organized as follows. In Section 2, we first recall some basic results about Carnot groups and their heat semigroups and then proceed to work out the optimal reverse Poincaré inequality. In Section 3, we show how the reverse Poincaré inequality implies the isoperimetric inequality. Finally in Section 4, we give a second application of the reverse Poincaré inequality, by showing that the Riesz transform on Carnot groups is bounded.

2 The optimal reverse Poincaré inequality for the heat semi-group in Carnot groups

2.1 Preliminaries on Carnot groups

A Carnot group of step (or depth) N is a simply connected Lie group \mathbb{G} whose Lie algebra can be written

$$\mathfrak{g} = \mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_N,$$

where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j}$$

and

$$\mathcal{V}_s = 0, \text{ for } s > N.$$

From the above properties, it is of course seen that Carnot groups are nilpotent. The number

$$Q = \sum_{i=1}^N i \dim \mathcal{V}_i$$

is called the homogeneous dimension of \mathbb{G} .

Example 2.1

1. (Commutative case) The group $(\mathbb{R}^d, +)$ is the only commutative Carnot group.
2. (Heisenberg groups) Consider the set $\mathbb{H}_n = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law

$$(x, \alpha) \star (y, \beta) = \left(x + y, \alpha + \beta + \frac{1}{2}\omega(x, y) \right),$$

where ω is the standard symplectic form on \mathbb{R}^{2n} , that is

$$\omega(x, y) = x^t \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix} y.$$

On \mathfrak{h}_n the Lie bracket is given by

$$[(x, \alpha), (y, \beta)] = (0, \omega(x, y)),$$

and it is easily seen that

$$\mathfrak{h}_n = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where $\mathcal{V}_1 = \mathbb{R}^{2n} \times \{0\}$ and $\mathcal{V}_2 = \{0\} \times \mathbb{R}$. Therefore \mathbb{H}_n is a Carnot group of depth 2.

3. (Engel group) The Engel group is the 4-dimensional Lie group of matrices

$$\mathbb{E} = \left\{ \begin{pmatrix} 1 & x & \frac{x^2}{2} & z \\ 0 & 1 & x & w \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}, x, y, w, z \in \mathbb{R} \right\}.$$

Its Lie algebra \mathfrak{e} is generated by the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

for which we have the following structure relations,

$$[X, Y] = W, \quad [X, W] = Z$$

and all other brackets are zero. The Engel group is easily seen to be a Carnot group of step 3.

On \mathfrak{g} we can consider the family of linear operators $\delta_t : \mathfrak{g} \rightarrow \mathfrak{g}$, $t \geq 0$ which act by scalar multiplication t^i on \mathcal{V}_i . These operators are Lie algebra automorphisms due to the grading. The maps δ_t induce Lie group automorphisms $\Delta_t : \mathbb{G} \rightarrow \mathbb{G}$ which are called the canonical dilations of \mathbb{G} . It is easily seen that there exists on \mathbb{G} a complete and smooth vector field D such that

$$\Delta_t = e^{(\ln t)D}.$$

This vector field D is called the dilation vector field on \mathbb{G} . If X is a left (or right) invariant smooth horizontal vector field on \mathbb{G} , that is $X(0) \in \mathcal{V}_1$, we have for every $f \in C^\infty(\mathbb{G})$, and $t \geq 0$,

$$X(f \circ \Delta_t) = tXf.$$

Let us now pick a basis V_1, \dots, V_d of the vector space \mathcal{V}_1 . The vectors V_i 's can be seen as left invariant vector fields on \mathbb{G} . In the sequel, these vector fields shall still be denoted by V_1, \dots, V_d and the corresponding right invariant vector fields shall be denoted by $\hat{V}_1, \dots, \hat{V}_d$. The left invariant sub-Laplacian on \mathbb{G} is the operator:

$$L = \sum_{i=1}^d V_i^2.$$

It is essentially self-adjoint on the space of smooth and compactly supported function $f : \mathbb{G} \rightarrow \mathbb{R}$ with respect to the Haar measure μ of \mathbb{G} . The heat semigroup $(P_t)_{t \geq 0}$ on \mathbb{G} , defined through the spectral theorem, is then seen to be a Markov semigroup (see [14]). This heat semigroup admits a positive fundamental solution named the heat kernel and denoted by $p_t(g, g')$. We often simply denote $p_t(g) = p_t(0, g)$ for the heat kernel issued from the identity. By left invariance, it is enough to know this heat kernel issued from the identity to recover all the heat kernels. Since Carnot groups enjoy the volume doubling property and carry the Poincaré inequality on balls, we deduce that p_t satisfies the double-sided Gaussian bounds (see Theorem 2.9 in [14]):

$$\frac{C^{-1}}{t^{Q/2}} \exp\left(-\frac{C_1 d(0, g)^2}{t}\right) \leq p_t(g) \leq \frac{C}{t^{Q/2}} \exp\left(-C_2 \frac{d(0, g)^2}{t}\right), \quad (2.2)$$

for some constants $C, C_1, C_2 > 0$. Here $d(0, g)$ is Carnot-Carathéodory distance from 0 to g in \mathbb{G} .

As usual, if $f : \mathbb{G} \rightarrow \mathbb{R}$ is a smooth function, we denote,

$$\Gamma(f, f) = \sum_{i=1}^d (V_i f)^2.$$

This is the *carré du champ* operator of L . Sometimes, as a shortened notation we will denote $\Gamma(f)$ for $\Gamma(f, f)$. The following gradient bound can be found in [14], Theorem 2.7,

$$\sqrt{\Gamma(p_t)}(g) \leq \frac{C}{t^{\frac{Q+1}{2}}} \exp\left(-C_3 \frac{d(0, g)^2}{t}\right). \quad (2.3)$$

We also introduce the right-invariant sub-Laplacian:

$$\hat{L} = \sum_{i=1}^d \hat{V}_i^2$$

and we denote by \hat{P}_t the associated heat semigroup and by \hat{p}_t the associated heat kernel.

We also denote

$$\hat{\Gamma}(f, f) = \sum_{i=1}^d (\hat{V}_i f)^2.$$

First, we begin with two useful lemmas, whose proofs are classical and let to the reader. The main argument is that for $f \in L^2(\mathbb{G}, \mu)$, $P_t f$ is the unique solution in $L^2(\mathbb{G}, \mu)$ of the parabolic Cauchy problem:

$$\begin{cases} \frac{\partial \phi}{\partial t} = L\phi \\ \phi(0, x) = f(x). \end{cases}$$

The same characterization holds for \hat{P}_t . Our first lemma relies the two semigroups P_t and \hat{P}_t .

Lemma 2.2 Let $f \in L^2(\mathbb{G}, \mu)$. Then for $g \in \mathbb{G}$, one has:

$$\hat{P}_t(f)(g) = P_t(f \circ Ad(g^{-1}))(g)$$

where $Ad(g^{-1})$ is the function defined by $Ad(g^{-1})(h) = g^{-1}hg$. As a consequence, one has:

$$\hat{P}_t(f)(0) = P_t(f)(0),$$

and for every $g \in \mathbb{G}$

$$\hat{p}_t(g) = p_t(g).$$

The second lemma illustrates the scaling property of the semigroup P_t with respect to the dilations. The second identity of the lemma can be obtained by differentiating the first one at $c = 1$.

Lemma 2.3 Let $f \in L^2(\mathbb{G}, \mu)$. For every $t, c \geq 0$,

$$P_t(f \circ \Delta_{\sqrt{c}}) = (P_{ct}f) \circ \Delta_{\sqrt{c}}.$$

Moreover, if $f \in C_0^\infty(\mathbb{G})$, then for every $t \geq 0$,

$$P_t Df = DP_t f + t P_t Lf.$$

We conclude this preliminary section with an integrability lemma which shall be useful in the sequel.

Lemma 2.4 Let $f : \mathbb{G} \rightarrow \mathbb{R}$ be a smooth function with polynomial growth, that is,

$$|f(g)| \leq C(1 + d(0, g))^N, \quad g \in \mathbb{G},$$

for some $C > 0, N \geq 0$, then for $t > 0$,

$$\int_{\mathbb{G}} f p_t \Gamma(\ln p_t, \ln p_t) d\mu < +\infty.$$

Proof. As the dilation vector field D vanishes at 0, for all $t > 0$ and for any smooth and compactly supported h ,

$$P_t((tL - D)h)(0) = 0,$$

that is

$$\int_{\mathbb{G}} (tL - D)h p_t d\mu = 0.$$

This implies

$$\left(tL + D + \frac{Q}{2} \right) p_t = 0$$

because the the adjoint D^* of D is $-D - \frac{Q}{2}$.

Let now $h \in C_0^\infty(\mathbb{G})$. We have

$$\begin{aligned} \int_{\mathbb{G}} h f p_t \Gamma(\ln p_t, \ln p_t) d\mu &= \int_{\mathbb{G}} h f \Gamma(\ln p_t, p_t) d\mu \\ &= \int_{\mathbb{G}} \Gamma(h f \ln p_t, p_t) d\mu - \int_{\mathbb{G}} \ln p_t \Gamma(h f, p_t) d\mu \\ &= - \int_{\mathbb{G}} h f \ln p_t L p_t d\mu - \int_{\mathbb{G}} \ln p_t \Gamma(h f, p_t) d\mu \\ &= - \frac{1}{t} \int_{\mathbb{G}} D(h f \ln p_t) p_t d\mu - \int_{\mathbb{G}} \ln p_t \Gamma(h f, p_t) d\mu \end{aligned}$$

In exponential coordinates, the vector fields D and V_i 's have polynomial coefficients. We can thus easily construct an increasing sequence $h_n \in C_0^\infty(\mathbb{G})$ such that $0 \leq h_n \leq 1$, $h_n \rightarrow 1$ and $|Dh_n|(g) \leq \frac{1}{n} P(g)$, $\sqrt{\Gamma}(h_n)(g) \leq \frac{1}{n} P(g)$, where P is a function with polynomial growth on \mathbb{G} . We now use the previous equalities with h_n in place of h . We have

$$\int_{\mathbb{G}} D(h_n f \ln p_t) p_t d\mu = \int_{\mathbb{G}} h_n D(f \ln p_t) p_t d\mu + \int_{\mathbb{G}} D(h_n) f p_t \ln p_t d\mu.$$

Thus, from the bounds (2.2) and (2.3), by dominated convergence, we obtain

$$\int_{\mathbb{G}} D(h_n f \ln p_t) p_t d\mu \rightarrow \int_{\mathbb{G}} D(f \ln p_t) p_t d\mu.$$

Similarly, we have

$$\int_{\mathbb{G}} \ln p_t \Gamma(h_n f, p_t) d\mu \rightarrow \int_{\mathbb{G}} \ln p_t \Gamma(f, p_t) d\mu.$$

□

2.2 The optimal reverse Poincaré inequality

We now turn to the inequality which is of interest for us. The following reverse Poincaré inequality for the heat semigroup is optimal.

Proposition 2.5 *Let $f : \mathbb{G} \rightarrow \mathbb{R}$ be a smooth and compactly supported function. For $g \in \mathbb{G}$,*

$$\Gamma(P_t f, P_t f)(g) \leq \frac{\Lambda}{t} (P_t f^2(g) - (P_t f)^2(g)), \quad t > 0.$$

where the constant Λ is the largest eigenvalue of the symmetric matrix

$$M = \left(\int_{\mathbb{G}} \hat{V}_i(\ln p_1) \hat{V}_j(\ln p_1) p_1 d\mu \right)_{1 \leq i, j \leq d}.$$

Moreover, the constant Λ is sharp.

Proof. By left invariance and scaling, it is enough to check it at the identity and $t = 1$. Let $f : \mathbb{G} \rightarrow \mathbb{R}$ be a smooth and compactly supported function, then

$$\Gamma(P_1 f, P_1 f)(0) = \sup_{\sum_{i=1}^d a_i^2 = 1} \left(\sum_{i=1}^d a_i V_i P_1(f)(0) \right)^2.$$

Now, let $a_i \in \mathbb{R}$ be such that $\sum_{i=1}^d a_i^2 = 1$, then

$$\begin{aligned} \sum_{i=1}^d a_i V_i P_1(f)(0) &= P_1 \left(\sum_{i=1}^d a_i \hat{V}_i f \right)(0) \\ &= - \int_{\mathbb{G}} \sum_{i=1}^d a_i \hat{V}_i(p_1) f d\mu \\ &= - \int_{\mathbb{G}} \sum_{i=1}^d a_i \hat{V}_i(\ln p_1) f p_1 d\mu \end{aligned}$$

Therefore, by Cauchy-Schwarz inequality against the measure $p_1 d\mu$, we have

$$\left(\sum_{i=1}^d a_i V_i P_1(f)(0) \right)^2 \leq \int_{\mathbb{G}} \left(\sum_{i=1}^d a_i \hat{V}_i(\ln p_1) \right)^2 p_1 d\mu \int_{\mathbb{G}} f^2 p_1 d\mu.$$

But one can write:

$$\begin{aligned} \int_{\mathbb{G}} \left(\sum_{i=1}^d a_i \hat{V}_i(\ln p_1) \right)^2 p_1 d\mu &= \sum_{i,j=1}^d a_i a_j \int_{\mathbb{G}} \hat{V}_i(\ln p_1) \hat{V}_j(\ln p_1) p_1 d\mu \\ &= A^t M A \end{aligned}$$

with M the matrix defined in the proposition and $A = (a_1, \dots, a_d)^t$. The result follows then easily

□

The following proposition gives a lower and upper bound for the optimal constant Λ .

Proposition 2.6 *We have*

$$\frac{Q}{2d} \leq \Lambda \leq \frac{Q}{2}.$$

Proof. Since the matrix M is symmetric, the constant Λ satisfies the following inequality:

$$\frac{1}{d} \operatorname{trace} M \leq \Lambda \leq \operatorname{trace} M$$

and the trace of M is given by

$$\operatorname{trace} M = \int_{\mathbb{G}} \hat{\Gamma}(\ln p_1) p_1 d\mu = \int_{\mathbb{G}} \Gamma(\ln p_1) p_1 d\mu.$$

Thus, we just need to prove that

$$\int_{\mathbb{G}} \Gamma(\ln p_t) p_t d\mu = \frac{Q}{2t}.$$

To this end, recall that

$$\left(tL + D + \frac{Q}{2} \right) p_t = 0.$$

Multiply then by $\ln p_t$ and taking integral gives

$$\int_{\mathbb{G}} \ln p_t \left(tL + D + \frac{Q}{2} \right) p_t d\mu = 0.$$

But by using an integration by parts (which we can justify as in Lemma 2.4), we have

$$t \int_{\mathbb{G}} \ln p_t L p_t d\mu = -t \int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) p_t d\mu.$$

Moreover, we have

$$\int_{\mathbb{G}} \ln p_t \left(D + \frac{Q}{2} \right) p_t d\mu = - \int_{\mathbb{G}} p_t D \ln p_t d\mu = - \int_{\mathbb{G}} D p_t d\mu = \frac{Q}{2} \int_{\mathbb{G}} p_t d\mu = \frac{Q}{2}$$

and therefore

$$\int_{\mathbb{G}} \Gamma(\ln p_t, \ln p_t) p_t d\mu = \frac{Q}{2t}.$$

□

2.3 The case of H-type groups

We now prove that the lower bound is achieved in a special class of Carnot groups, the so-called *H*-type groups. For us, a *H*-type group will be simply $\mathbb{R}^{2n+m} = \mathbb{R}^{2n} \times \mathbb{R}^m$ equipped with the product

$$v * w = v + w + \frac{1}{2}[v, w]$$

where $[\cdot, \cdot]$ is a bracket operation on \mathbb{R}^{2n+m} whose center is $0 \times \mathbb{R}^m$ and such the map $J_z : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ defined for $z \in \mathbb{R}^m$ by:

$$\langle J_z(x), y \rangle = \langle [x, y], z \rangle \text{ for all } x, y \in \mathbb{R}^{2n}$$

is orthogonal when $|z| = 1$. In the above we have identified $x \in \mathbb{R}^{2n}$ with $(x, 0) \in \mathbb{R}^{2n} \times \mathbb{R}^m$ and $z \in \mathbb{R}^m$ with $(0, z) \in \mathbb{R}^{2n} \times \mathbb{R}^m$ and $|\cdot|$ denotes the classical Euclidean norm.

The left-invariant vector fields which coincide with $\frac{\partial}{\partial x_i}$ at the identity write:

$$V_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{j=1}^m \langle J_{u_j} x, e_i \rangle \frac{\partial}{\partial z_j}$$

whereas the corresponding right-invariant vector fields write:

$$\hat{V}_i = \frac{\partial}{\partial x_i} - \frac{1}{2} \sum_{j=1}^m \langle J_{u_j} x, e_i \rangle \frac{\partial}{\partial z_j}$$

for $i = 1, \dots, 2n$ and where (e_1, \dots, e_{2n}) is the canonical basis of \mathbb{R}^{2n} and (u_1, \dots, u_m) the canonical basis of \mathbb{R}^m . It is easy to see that it is a Carnot group of step 2. The Haar measure is just the Lebesgue measure and the heat kernel associated to $L = \sum_{i=1}^{2n} V_i^2$ issued from the identity is given by (see for instance [8]):

$$p_t(x, z) = \frac{1}{(2\pi)^m} \frac{1}{(4\pi)^n} \int_{\mathbb{R}^m} e^{i \langle \lambda, z \rangle} e^{-\frac{|\lambda||x|^2}{4}} \coth |\lambda|t \left(\frac{|\lambda|}{\sinh |\lambda|t} \right)^n d\lambda$$

and therefore is only a function of $|x|$ and $|z|$.

In the case of H -type groups, we obtain the more precise statement.

Corollary 2.7 *Assume \mathbb{G} to be an H -type group, then*

$$\Lambda = \frac{Q}{2d}.$$

Proof. Let \mathbb{G} be a H -type group. By the previous proposition, the only thing that we need to do is to see that, in this case, the matrix M writes λId for some $\lambda \in \mathbb{R}$. This will come from the fact that the heat kernel is a radial function, this means that it only depends on $|x|$ and $|z|$. Recall also that a H -type group can be identified with \mathbb{R}^{2n+m} and that the vector fields V_i , $i = 1, \dots, 2n$ read:

$$\hat{V}_i = \frac{\partial}{\partial x_i} - \frac{1}{2} \sum_{j=1}^m \langle J_{u_j} x, e_i \rangle \frac{\partial}{\partial z_j}$$

where J_{u_j} is an orthogonal map of \mathbb{R}^{2n} . Therefore the coefficient M_{ij} of the matrix M can be written as:

$$\int_{x \in \mathbb{R}^{2n}} \int_{z \in \mathbb{R}^m} \hat{V}_i f(|x|^2, |z|^2) \hat{V}_j f(|x|^2, |z|^2) g(|x|^2, |z|^2) dx dz$$

for some functions f and g . Now,

$$V_i f(|x|^2, |z|^2) = 2x_i \partial_1 f - \sum_{l=1}^m z_l \langle J_{u_l} x, e_i \rangle \partial_2 f$$

and the result is coming by expanding the product and noticing that:

$$\int_{x \in \mathbb{R}^{2n}} x_i x_j h_1(|x|, |z|) dx = \delta_{ij} \tilde{h}_1(|z|),$$

$$\int_{z \in \mathbb{R}^m} z_j h_2(|x|, |z|) dz = 0,$$

$$\int_{z \in \mathbb{R}^m} z_l z_p h_3(x) = 0 \text{ for } l \neq p,$$

$$\int_{x \in \mathbb{R}^{2n}} z_l^2 < J_{u_l} x, e_i > < J_{u_l} x, e_j > h_4(|x|, |z|) dx = \delta_{ij} z_l^2 \tilde{h}_4(|z|)$$

since $J_{u_l} x$ is an orthogonal map of \mathbb{R}^{2n} for $l = 1, \dots, m$ and that $\int_{z \in \mathbb{R}^m} z_l^2 \tilde{h}_4(|z|) dz$ does not depend on l .

Note that the estimates in [8] show that all the integrals appearing in the above argument are well defined. \square

3 Isoperimetric inequality

In this section, we show that the reverse Poincaré inequality we proved in the previous section can be used to prove the isoperimetric inequality in Carnot groups. To this end, we adapt some beautiful ideas of Varopoulos (see [13], pp.256-258) and Ledoux (see pp. 22 in [10], see also Theorem 8.4 in [11])

In order to state the inequality we need the notion of horizontal perimeter, which is defined from the horizontal variation of a function.

Following [6], given a function $f \in L^1_{loc}(\mathbb{G})$ we define the horizontal total variation of f by

$$\text{Var}_{\mathcal{H}}(f) = \sup_{\phi \in \mathcal{F}(\mathbb{G})} \int_{\mathbb{G}} f \left(\sum_{i=1}^d V_i \phi_i \right) d\mu.$$

where

$$\mathcal{F}(\mathbb{G}) = \{\phi \in C_0^1(\mathbb{G}, \mathcal{H}) \mid \|\phi\|_{\infty} \leq 1\}.$$

Here, for $\phi = \sum_{i=1}^d \phi_i V_i$, we have let $\|\phi\|_{\infty} = \sup_{\mathbb{G}} \sqrt{\sum_{i=1}^d \phi_i^2}$.

The space

$$BV_{\mathcal{H}}(\mathbb{G}) = \{f \in L^1(\mathbb{G}) \mid \text{Var}_{\mathcal{H}}(f) < \infty\},$$

endowed with the norm

$$\|f\|_{BV_{\mathcal{H}}(\mathbb{G})} = \|f\|_{L^1(\mathbb{G})} + \text{Var}_{\mathcal{H}}(f),$$

is a Banach space. One can note that when $f \in W_{\mathcal{H}}^{1,1}(\mathbb{G})$, then $f \in BV_{\mathcal{H}}(\mathbb{G})$, and one has in fact

$$\text{Var}_{\mathcal{H}}(f) = \|\sqrt{\Gamma(f)}\|_{L^1(\mathbb{G})}.$$

Given a measurable set $E \subset \mathbb{G}$ we say that it has finite horizontal perimeter if $\mathbf{1}_E \in BV_{\mathcal{H}}(\mathbb{G})$. In such case we define the horizontal perimeter of E by

$$P_{\mathcal{H}}(E) = \text{Var}_{\mathcal{H}}(\mathbf{1}_E).$$

We say that a measurable set $E \subset \mathbb{G}$ is a Caccioppoli set if $P_{\mathcal{H}}(E) < \infty$.

We now prove the following result which is due to Varopoulos.

Theorem 3.1 (Isoperimetric inequality) *There is a constant $C_{\text{iso}} > 0$, such that for every bounded Caccioppoli set $E \subset \mathbb{G}$*

$$\mu(E)^{\frac{Q-1}{Q}} \leq C_{\text{iso}} P_{\mathcal{H}}(E).$$

Proof. Let $f \in C_0^\infty(\mathbb{G})$. From Proposition 2.5, we have

$$\Gamma(P_t f) \leq \frac{\Lambda}{t} \|f\|_{L^\infty(\mathbb{G})}^2, \quad t > 0.$$

Thus,

$$\|\sqrt{\Gamma(P_t f)}\|_{L^\infty(\mathbb{G})} \leq \sqrt{\frac{\Lambda}{t}} \|f\|_{L^\infty(\mathbb{G})}.$$

Applying this inequality to $g \in C_0^\infty(\mathbb{G})$, with $g \geq 0$ and $\|g\|_{L^\infty(\mathbb{G})} \leq 1$, if $f \in C_0^1(\mathbb{G})$ we have

$$\begin{aligned} \int_{\mathbb{G}} g(f - P_t f) d\mu &= \int_0^t \int_{\mathbb{G}} g \frac{\partial P_s f}{\partial s} d\mu ds = \int_0^t \int_{\mathbb{G}} g L P_s f d\mu ds = - \int_0^t \int_{\mathbb{G}} \Gamma(P_s g, f) d\mu ds \\ &\leq \int_0^t \|\sqrt{\Gamma(P_s g)}\|_{L^\infty(\mathbb{G})} \int_{\mathbb{G}} \sqrt{\Gamma(f)} d\mu ds \leq 2 \sqrt{\Lambda t} \int_{\mathbb{G}} \sqrt{\Gamma(f)} d\mu. \end{aligned}$$

We thus obtain the following basic inequality: for $f \in C_0^1(\mathbb{G})$,

$$\|P_t f - f\|_{L^1(\mathbb{G})} \leq 2 \sqrt{\Lambda t} \|\sqrt{\Gamma(f)}\|_{L^1(\mathbb{G})}, \quad t > 0. \quad (3.4)$$

Suppose now that $E \subset \mathbb{G}$ is a bounded Caccioppoli set. Therefore $\mathbf{1}_E \in BV_{\mathcal{H}}(\mathbb{G})$. By Theorem 1.14 in [9]. there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C_0^\infty(\mathbb{G})$ satisfying

- (i) $\|f_n - \mathbf{1}_E\|_{L^1(\mathbb{G})} \rightarrow 0$;
- (ii) $\int_{\mathbb{G}} \sqrt{\Gamma(f_n)} d\mu \rightarrow \text{Var}_{\mathcal{H}}(f)$.

Applying (3.4) to f_n we obtain

$$\|P_t f_n - f_n\|_{L^1(\mathbb{G})} \leq 2 \sqrt{\Lambda t} \|\sqrt{\Gamma(f_n)}\|_{L^1(\mathbb{G})} = 2 \sqrt{\Lambda t} \text{Var}_{\mathcal{H}}(f_n), \quad n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ in this inequality, we conclude

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(\mathbb{G})} \leq 2 \sqrt{\Lambda t} \text{Var}_{\mathcal{H}}(\mathbf{1}_E) = 2 \sqrt{\Lambda t} P_{\mathcal{H}}(E), \quad t > 0.$$

Observe now that we have

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(\mathbb{G})} = 2 \left(\mu(E) - \int_E P_t \mathbf{1}_E d\mu \right).$$

On the other hand, we have

$$\int_E P_t \mathbf{1}_E d\mu = \int_{\mathbb{M}} (P_{t/2} \mathbf{1}_E)^2 d\mu.$$

We thus obtain

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(\mathbb{G})} = 2 \left(\mu(E) - \int_{\mathbb{G}} (P_{t/2} \mathbf{1}_E)^2 d\mu \right).$$

We now note that

$$\begin{aligned} \int_{\mathbb{G}} (P_{t/2} \mathbf{1}_E)^2 d\mu &\leq \left(\int_E \left(\int_{\mathbb{G}} p_{t/2}(x, y)^2 d\mu(y) \right)^{\frac{1}{2}} d\mu(x) \right)^2 \\ &= \left(\int_E p_t(x, x)^{\frac{1}{2}} d\mu(x) \right)^2 \leq \frac{p_1(0)}{t^{Q/2}} \mu(E)^2. \end{aligned}$$

Combining these equations yields

$$\mu(E) \leq \sqrt{\Lambda t} P_{\mathcal{H}}(E) + \frac{p_1(0)}{t^{Q/2}} \mu(E)^2, \quad t > 0.$$

Minimizing in t , we conclude

$$\mu(E)^{\frac{Q-1}{Q}} \leq C P_{\mathcal{H}}(E),$$

with

$$C = (1+Q)^{\frac{Q+1}{Q}} p_1(0)^{\frac{1}{Q}} \frac{\Lambda}{Q}.$$

□

4 Boundedness of the Riesz transform

Besides the isoperimetric inequality, the reverse Poincaré for the heat semigroup is also intimately related to the boundedness of the Riesz transform. Actually combining Proposition 2.5 with the results in [5] which built on [1] leads to the following result.

Proposition 4.1 *Let $1 < p < \infty$. There exist constants $A_p, B_p > 0$ such that*

$$A_p \|(-L)^{1/2} f\|_p \leq \|\sqrt{\Gamma(f)}\|_p \leq B_p \|(-L)^{1/2} f\|_p, \quad f \in C_0^\infty(\mathbb{G}), \quad (4.5)$$

Proof. The main ingredient is Theorem 1.3 in [1]. The theorem is stated for Riemannian manifolds, but it is checked in [5] that the arguments go through in the context of Carnot-Carathéodory spaces that satisfy the doubling condition and the Poincaré inequality. The only thing to check is the bound

$$\|\sqrt{\Gamma} e^{tL}\|_{\infty \rightarrow \infty} \leq \frac{C}{\sqrt{t}},$$

where e^{tL} is the heat semigroup generated by L . This bound is a consequence of our Proposition 2.5. \square

We note that the first proof of the boundedness of the Riesz transform in Carnot groups can be found in [12].

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